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NGR-10-007-010

A STATISTICAL APPROACH TO THE THEORY OF INTERFACIAL CONDUCTIVITY I

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July 1967

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* This research is supported by the National Aeronautics and Space Administration Contract No. NASA NGR 10-007-010 and No. NAS8-20126.

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Abstract:

A simple model of the contact between two surfaces is given, taking into account the statistical features of this contact. A theory of interfacial thermal (or electrical) conductivity is based on this model. The information that enters our theory is the height distribution of the asperities on the surfaces. For some simple distributions the heat flow is calculated as a function of the applied pressure, taking into account elastic as well as inelastic deformations of the surfaces.

For more realistic height distributions the required integrals can easily be evaluated numerically.

1. Introduction

One of the most complicated tasks in any calculation of the interfacial thermal conductivity (ITC) is the description of a rough surface. Various statistical [1-3] and semi-empirical [4] approaches have been proposed to describe the properties of a surface and the contact between two such surfaces.

In this paper we shall investigate a special model of a surface. This model can best be applied to milled surfaces, since their characteristics resemble most closely the assumptions made here. This restriction to a special class of surfaces enables us to find rather explicit expressions for the heat flow, surface area in contact, etc. Furthermore, inelastic and elastic contacts can be taken into account simultaneously. Various other generalizations of the model, i.e. inclusion of void phase conduction, effects of surface films, etc. are discussed in Chapter 9 and will be studied further in parts II and III of this paper.

Here we shall restrict ourselves first to the simplest possible statistical theory of the contact between two surfaces, in order to present the essential features of the theory clearly.

2. The Model

The surface of the metal will be represented by a series of equidistant spherical asperities of varying height, but equal curvature. Fig. 1 shows a cross section through our surface model.

Fig. 1

Each spherical asperity is thus contained in a square of area B^2. The distribution of the heights of the hills will be described in a statistical sense by a function $\rho(h)$ such that $\rho(h)dh$ = probability that an asperity has a height between h and $h + dh$. (1)

ρ has to be normalized such that

$$\int \rho\,(h)\,dh = 1 \tag{2}$$

We shall always assume that there is an upper limit to the height of the hills, i.e. $\rho\,(h) = o \quad h > H$.

If we place two such surfaces above one another as shown in Fig. 2, they will begin to touch as soon as their distance of approach L reaches $2H$.

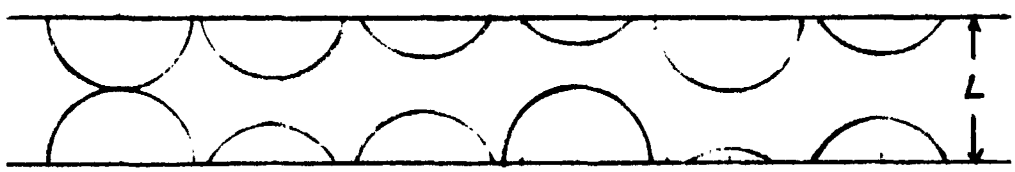

Fig. 2

However, only very few contact points will touch in this case. With rising pressure L decreases and the number of contact points increases accordingly. At the same time the contact points will be deformed elastically or inelastically and ITC sets in.

An essential drawback of our model is, of course, that we assume the asperities to be distributed regularly. Furthermore, we assume that the hills on the two surfaces oppose one another exactly and no contacts such as the one in Fig. 3 are considered.

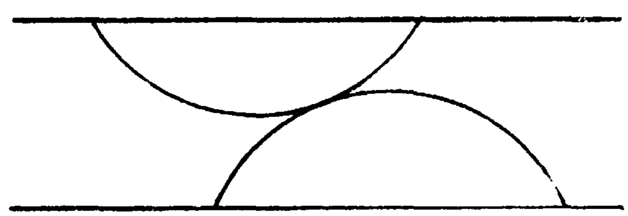

Fig. 3

This limits the applicability of our theory essentially to the contact of two milled surfaces. However, many of the results presented here are probably true also for other types of surface finish.

In the sequel we shall call opposing squares (containing one asperity each) a contact square, irrespective of the fact whether the two spheres actually touch or not.

3. Review of the Results for a Single Contact Point

The elastical deformation of two spheres that touch one another has been calculated by H. Hertz [5] . The basic results (generalizations of which have been given in [6] are the following). See Fig. 4.

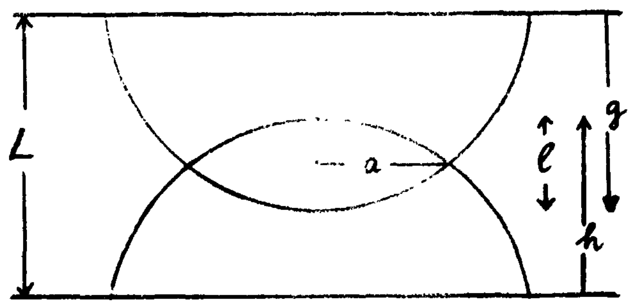

Fig. 4

If the two spheres are pressed together with a total force F they will be deformed elastically such that their apparent penetration depth ℓ is given by

$$\ell = (2F^2 D^2 / R)^{1/3} \tag{3}$$

where

$$D = \frac{3}{4} \frac{1 - \sigma^2}{E} \tag{4}$$

(we assume that both spheres consist of the same material). The radius of the contact circle is given by

$$a = \left(\frac{FRD}{2}\right)^{1/3} \tag{5}$$

The standard treatment of the heat flow through such a contact point is due to Cetinkale (Veziroglu) [7].

The contact is replaced by an idealized contact element as shown in Fig. 5.

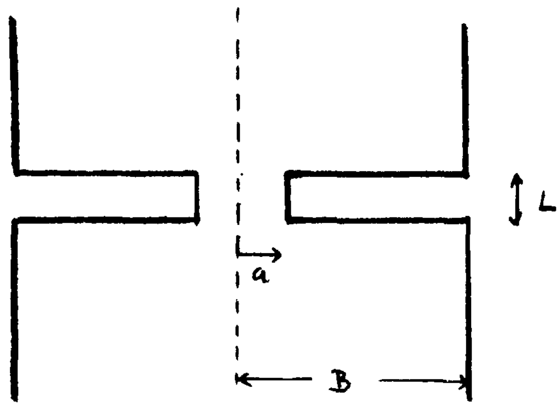

Fig. 5

The solution of the boundary value problem for the heat flow is rather complicated, due to the presence of liquid (lubricant) or gas in the gap between the two surfaces.

Since we are interested here mainly in the statistical features of the theory we shall consider the simplest case first, i.e. contact in vacuum. In this case the heat flow h becomes

$$h_c = 2\,a\,k\,\Delta\,T \tag{6}$$

where ΔT is the temperature step as the interface and k is the thermal conductivity of the material. It is perhaps somewhat surprising that the heat flow is proportional to a rather than a^2 and independent of the height of the contact element (Equation (6)) is valid for $L \ll a$, $B \gg a$).

The behavior of h is, however, due to the fact that the heat flow converges at the interface and is concentrated at the edges of the circle of radius a. This "skin effect" causes the dependence on a rather than a^2, in complete analogy to the usual electromagnetic skin effect.

Equations (3-6) are the starting point of our theory.

4. Statistical Treatment of a Contact Element: Elastic Deformations

Consider the situation shown in Fig. 4, where two lumps of heights h and g, respectively, have penetrated a distance ℓ given by

$$h + g = L + \ell \tag{7}$$

Eliminating F from equations (3) and (5) we obtain for the radius of the contact point

$$a = \sqrt{\frac{\ell R}{2}} \tag{8}$$

and the heat flow through this contact point becomes

$$h_c = 2k \Delta T \sqrt{\frac{R}{2}} \cdot \sqrt{\ell} \equiv \alpha \sqrt{\ell} \tag{9}$$

The penetration depth ℓ will, of course, differ for various contacts, because of the statistical distribution of their heights. We shall calculate now the average heat flux through one of the contact elements (keeping the distance L between the two surfaces fixed).

Consider first a sphere of fixed height g above the lower surface (see Fig. 4). Then the heat flow through the contact element will depend on the height of the upper sphere, i.e.

$$h_c = \alpha\sqrt{\ell} = \alpha\sqrt{h + g - L} \tag{10}$$

Since these heights h have a statistical distribution $\rho(h)$, the average heat flow (averaged over many contacts with varying h) will be

$$h_c \text{ (averaged over } h) = \alpha \int_{L-g}^{H} (h + g - L)^{\frac{1}{2}} \rho(h)\, dh$$

$$\langle h_c \rangle = \alpha \int_{0}^{H} dg\rho(g) \int_{L-g}^{H} dh\rho(h)\,(h + g - L)^{\frac{1}{2}} \tag{11}$$

This gives the average heat flux $\langle h_c \rangle$ through one contact element as a function of L.

To obtain the total heat flux through a surface of area A we have to multiply with the number N of contact square, i.e. with $N = A/B^2$.

Similarly we calculate now the average force on the two interfaces as a function of L. From (3) we have

$$F = \gamma\, \ell^{3/2} \qquad \gamma = \frac{1}{D}\sqrt{\frac{R}{2}} \tag{12}$$

The average force due to one contact point becomes

$$\langle F \rangle = \gamma \int (h + g - L)^{3/2} \quad \rho(h) \quad \rho(g) \; d\,h\,d\,g \tag{13}$$

which together with

$$\langle h_c \rangle = \alpha \int (h + g - L)^{1/2} \quad \rho(h) \quad \rho(g) \; d\,h\,d\,g \tag{14}$$

determines the heat flow as a function of the applied pressure $P = \langle F \rangle / B^2$.

The integrals (13) and (14) can be extended over a region in the (g,h) plan, such that $h + g > L$, i.e. we have to average only over such contact squares that actually touch one another. This region of integration is shown in Fig. 6.

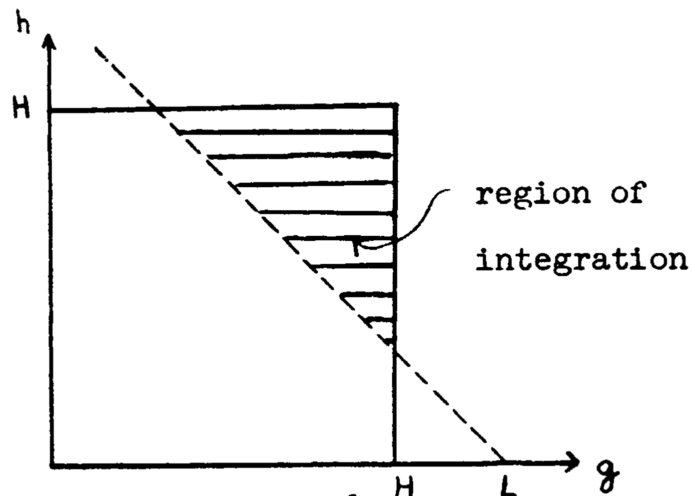

Fig. 6

5. Statistical Treatment of a Contact Element: Inelastic Deformations

If a critical pressure p_c (equal to the hardness of the material) is exceeded the material begins to flow plastically. While it is hardly possible to treat these phenomena exactly we shall take them into account in the phenomenological manner described below.

In this treatment we shall assume that in the case of plastic flow the basic relations (6) and (8) remain valid, while the relation (5) between the radius of the contact point and the force has to be changed. The simplest modification of (5) is

$$F = p_c \ \pi \ a^2 \tag{15}$$

i.e. the material flows until the pressure is equal to the critical pressure all over the contact surface. Inserting (8) this becomes

$$F = p_c \ \pi \ \frac{\ell \ R}{2} \tag{16}$$

This relation replaces (12) in the case of inelastic contacts, while the expression for the heat flux h_c remains unchanged.

Next we have to determine when the critical pressure will be exceeded at a contact point. The pressure distribution at the contact is (see e.g. [6])

$$p = \frac{3 \ F}{2\pi \ a^2} \ \sqrt{1 - r^2/a^2} \tag{17}$$

The pressure is thus largest in the center

$$p_o = \frac{3 \ F}{2\pi \ a^2} \tag{18}$$

Plastic flow sets in if $p_o > p_c$. Since both F and a are functions of $\ell = h + g - L$, this means that there is a critical value of ℓ, $\ell = \ell_c$, above which the contact begins to flow.

Therefore, the situation is as shown in Fig. 7.

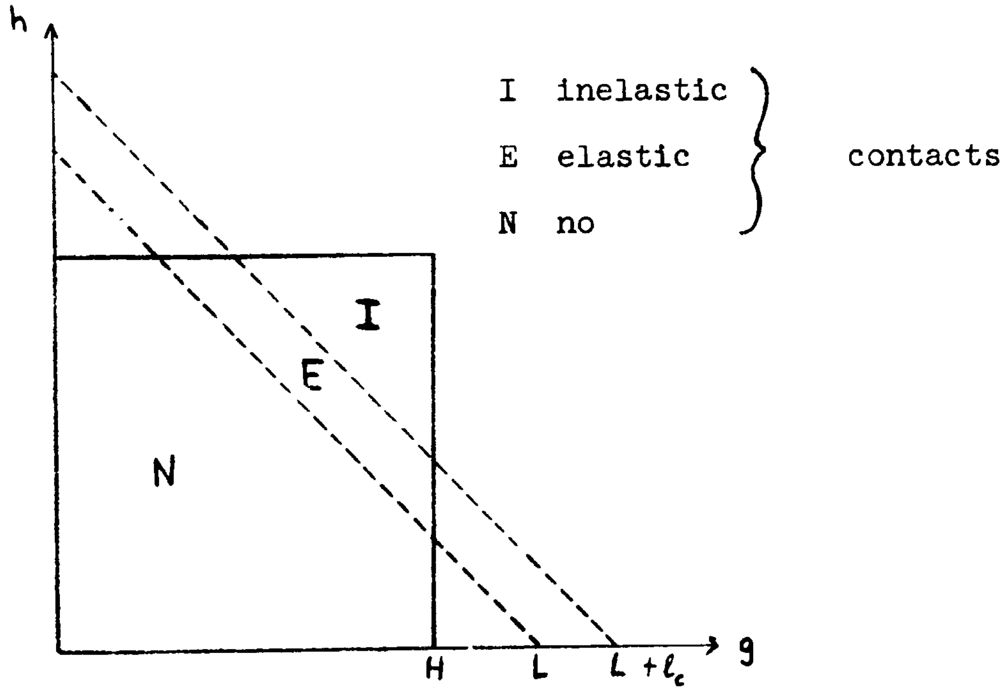

Fig.7

Thus the relation between the force F and the surface distance L is modified to

$$\langle F\rangle = \gamma \int_E (h + g - L)^{3/2} \rho(h) \quad \rho(g) \quad dh\, dg \; + \\ + \frac{p_c\ \pi}{2} \quad R \int_I (h + g - L) \quad \rho(h) \quad o(g) \quad dh\, dg \tag{19}$$

where the integrals are to be extended over the elastic (E) and inelastic (I) regions, respectively.

The relation between p_c and ℓ_c is given by

$$p_c = \frac{3}{\pi} \; \frac{1}{D} \sqrt{\frac{\ell_c}{R}} \tag{20}$$

This follows from (18), (12) and (8).

Equations (14) and (19) are our basic results; they determine heat flow and force as functions of the distance of approach.

6. The Interface Correlation Function

The two-dimensional integrals that appear in (14) and (19) can be re-written by a transformation of variables into a considerably simpler form.

The integrals that appear in (14) and (19) are of the general type

$$A_n(L) = \int (h + g - L)^n \ \rho(h) \ \rho(g) \ dh \, dg \tag{21}$$

integrated over some suitable domain. We introduce new variables by

$$\begin{aligned} h &= x + y \qquad & h + g &= 2x \\ g &= x - y \qquad & h - g &= 2y \end{aligned} \tag{22}$$

A_n (L) then becomes

$$\begin{aligned} A_n(L) &= 2 \iint (2x - L)^n \ \rho(x + y) \ \rho(x - y) \ d\,x\,dy = \\ &= 2\int (2x - L)^n \ K(x) \ dx \end{aligned} \tag{23}$$

where

$$K(x) = \int dy \ \rho(x + y) \ \rho(x - y) \tag{24}$$

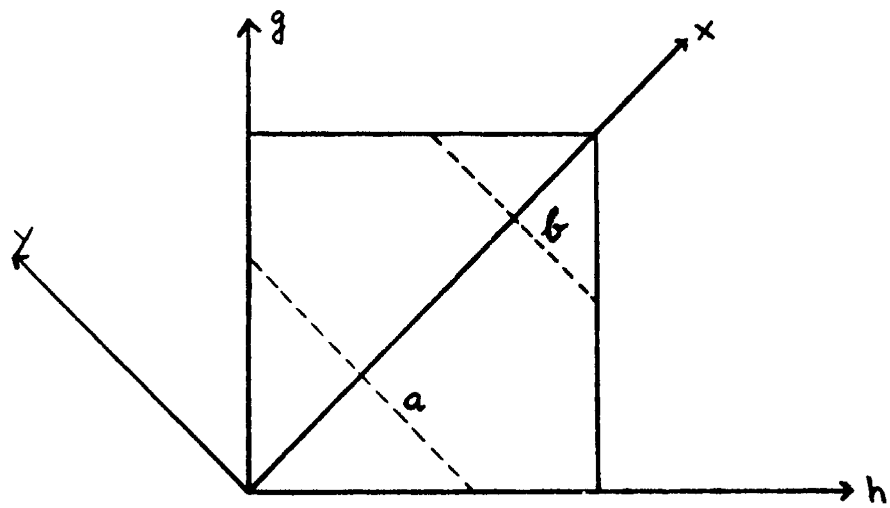

Fig. 8

The integral (24) is to be taken along the dotted lines (Fig. 8) in the (x,y) plane. Therefore the limits of integration are

$$K(x) = \int_{-x}^{x} dy \; \rho(x+y) \; \rho(x-y) \tag{25}$$

if the path is analogous to (a) in Fig. 8 or

$$K(x) = 2\int_{0}^{H-x} dy \; \rho(x+y) \; \rho(x-y) \tag{26}$$

for paths like (b). In the last step we used the obvious symmetry of the integrand of (26).

For simplicity we shall restrict ourselves here to case (b), i.e. we do not consider distance of approach $L < H$. This can, however, easily be done if necessary.

K(x) will be called the interface correlation function. It plays the dominant role in the theory of the contact between two surfaces. To bring our result into its final form we introduce a new variable

$$z = 2x - L = h+g - L \tag{27}$$

in (24). The integrals A_n there become

$$A_n(L) = \int z^n K(\tfrac{z+L}{2}) \; dz \tag{28}$$

The limits of integration over z still have to be determined.

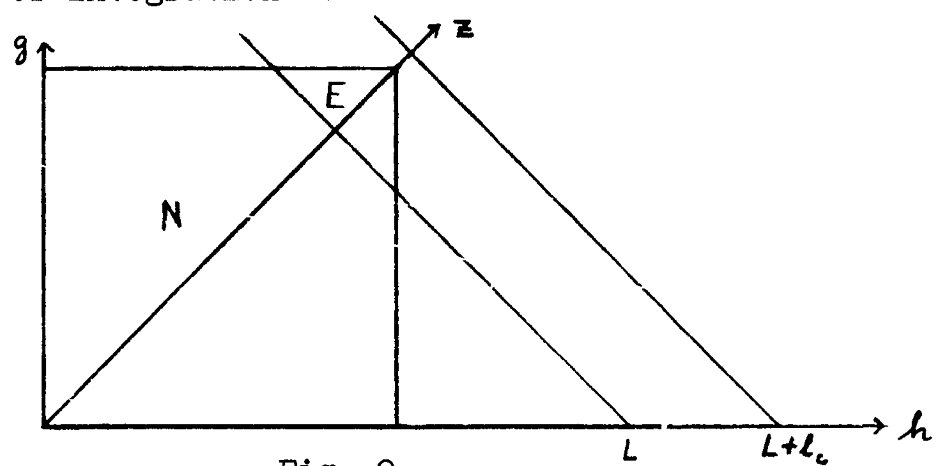

Fig. 9

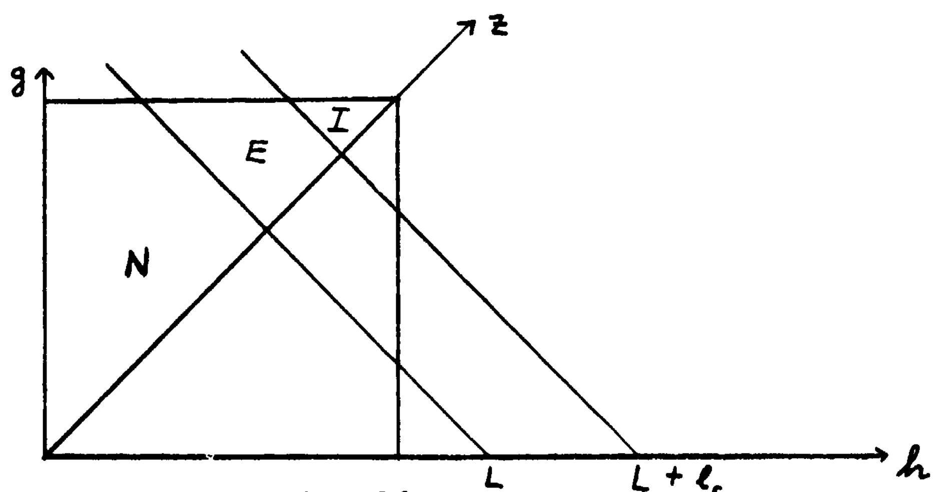

Fig. 10

Let us first study the case of elastic contacts only. Then the limits of the integration are given by $z = 0$ and $z = 2H - L = d$ respectively, as can easily be seen from Fig. 9.

In the presence of elastic and inelastic contact points the situation is as shown in Fig. 10. There the elastic integrals go from 0 to ℓ_c, the inelastic ones from ℓ_c to d. Therefore, the expression for heat flux $< h_c >$ and force become finally

$$\langle h_c \rangle = \alpha \int_0^d \sqrt{z} \quad K\left(\frac{z + L}{2}\right) \; dz \tag{29}$$

$$\langle F \rangle = \begin{cases} \gamma \displaystyle\int_0^d z^{3/2} \; K\left(\frac{z + L}{2}\right) \; dz & \underline{\text{Elastic contacts only}} \\ \gamma \displaystyle\int_0^{\ell_c} z^{3/2} \; K\left(\frac{z + L}{2}\right) \; dz + \frac{p_c \, T \, R}{2} \displaystyle\int_{\ell_c}^d z \; K\left(\frac{z + L}{2}\right) \; dz & \end{cases} \tag{30}$$

<u>Elastic and inelastic contacts</u>

where

$$\alpha = 2k \, \Delta \, T \sqrt{\frac{R}{2}} \qquad \gamma = \frac{1}{D}\sqrt{\frac{R}{2}} \qquad D = \frac{3}{4} \left(\frac{1 - \sigma^2}{E}\right)$$

$$d = 2H - L \qquad p_c = \frac{3}{\pi} \, \frac{1}{D} \sqrt{\frac{\ell_c}{R}} \tag{31}$$

$$K(x) = 2 \int_0^{H - x} dy \; \rho(x + y) \; \rho(x - y)$$

Equations (29-31) are our final results.

7. Examples

We shall calculate how the expectation values for the heat flux and the force F for some simple distribution functions.

The first step is the calculation of the interface correlation function.

The following height distribution functions will be considered

$$\begin{aligned} &\text{a)} \quad \rho_\delta \ (x) = \delta\ (H - x) \\ &\text{b)} \quad \rho_\theta \ (x) = \frac{1}{H}\ \theta\ (H - x) \\ &\text{c)} \quad \rho_m \ (x) = (H - x)^m \ v_m \end{aligned} \tag{32}$$

(v_m is a normalization factor)

The corresponding interface correlation functions are

$$\begin{aligned} &\text{a)} \quad K(x) = \delta\ (H - x) \\ &\text{b)} \quad K(x) = \frac{2}{H^2}(H - x) \qquad o \le x \le H \ , \\ &\text{c)} \quad K(x) = w_m (H - x)^{2m+1} \qquad o \le x \le H \ , \end{aligned} \tag{33}$$

(w_m = normalization factor)

Inserting these correlation functions into the integrals (29) and (30) respectively, we obtain the following expressions for the heat flux and force (we restrict ourselves at the moment to elastic contacts)

$$
\begin{aligned}
&\text{a)} \quad \langle h_c \rangle \simeq d^{\frac{1}{2}} \\
&\qquad \langle F \rangle \simeq d^{\frac{3}{2}} \\
&\text{b)} \quad \langle h_c \rangle = \frac{4}{15} \frac{d^{\frac{5}{2}}}{H^2} \\
&\qquad \langle F \rangle = \frac{4\gamma}{35H^2} \cdot d^{\frac{7}{2}} \\
&\text{c)} \quad \langle h_c \rangle \simeq d^{\frac{5}{2} + 2m} \\
&\qquad \langle F \rangle \simeq d^{\frac{7}{2} + 2m}
\end{aligned} \tag{34}
$$

Therefore, simple power laws result

$$
\begin{aligned}
&\text{a)} \quad h_c \simeq F^{\frac{1}{3}} \\
&\text{b)} \quad h_c \simeq F^{\frac{5}{7}} \\
&\text{c)} \quad h_c \simeq F^{\frac{5+4m}{7+4m}}
\end{aligned} \tag{35}
$$

(35a) is just the well known relation between heat flow and force that results from the assumption that all asperities have the same height H.

As an example for the transition from elastic to inelastic deformations we calculate the heat flow in case b, the result is

$$\langle h_c \rangle = \frac{4\alpha}{15} \frac{d^{\frac{5}{2}}}{H^2}$$

$$\langle F \rangle = \begin{cases} \frac{4\gamma}{35H^2} \, d^{\frac{7}{2}} & d < \ell_c \\ \frac{2\gamma}{35H^2} \, \ell_c^{\frac{5}{2}} \, (7d - 5\ell_c) + \frac{\pi p c R}{12H^2} \, (d - \ell_c)^2 \, (d + 2\ell_c) & d > \ell_c \end{cases} \quad (36)$$

Here the power law changes slowly from

$$h_c \sim F^{\frac{5}{7}} \qquad \text{to} \qquad h_c \sim F^{\frac{5}{6}} \quad (37)$$

when inelastic effects set in.

The remarkable common feature of all the examples given above is that they result in power laws with exponents smaller than one. In this respect our results are similar to those of Held [4] .

For more complicated height distribution function the heat flow can be calculated numerically only.

The strong dependence of the functional form of $h_c(F)$ on the height distribution indicates why attempts to calculate the heat flow from simple models are generally unsuccessful.

The results obtained hitherto are in agreement with the experimental data in the low and medium pressure region ($p < 10^{-3}\ p_c$) where power laws with exponents < 1 are generally observed. At higher pressures, however, a much stronger increase of the heat flow is sometimes observed, suggesting power laws $h_c \sim F^2$. The standard explanation of this is that in this pressure region a strong increase in the number of contact points takes place. A quantitative formulation of this argument can be given in our theory, by assuming a height distribution of the form

$$\rho\,(h) = \frac{s}{H}\;\theta\,(H - x) + (1 - s)\;\delta\,(G - x) \qquad G < H \quad s < 1 \tag{38}$$

A typical curve $h_c(F)$ that results from (38) is shown in Fig. 11 (assuming inelastic contacts only).

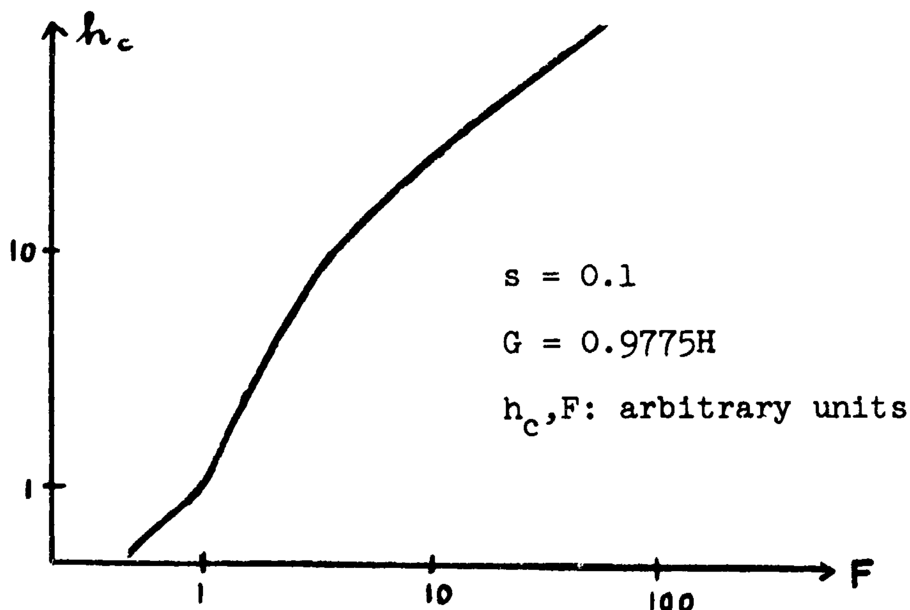

Fig. 11

The onset of the change in slope at F_1 depends, of course, on the values of the parameters used in (38), as does F_2. With typical parameters s and G one obtains $F_2/F_1 \sim 2 - 4$, i.e. the steep slope can be maintained only over a relatively small pressure range. Since this is not in agreement with the experiment, the slopes > 1 will have to be interpreted in a different way. This will be done in Chapter 9.

8. Surface Films and Electrical Conductivity

It has often been suggested that the mechanism for interfacial thermal and electrical conductivity are rather similar. Recent measurements by Fried [8] have shown that this is only partially correct. A typical plot of the ratio electrical/thermal resistance vs. pressure is shown in Fig. 12.

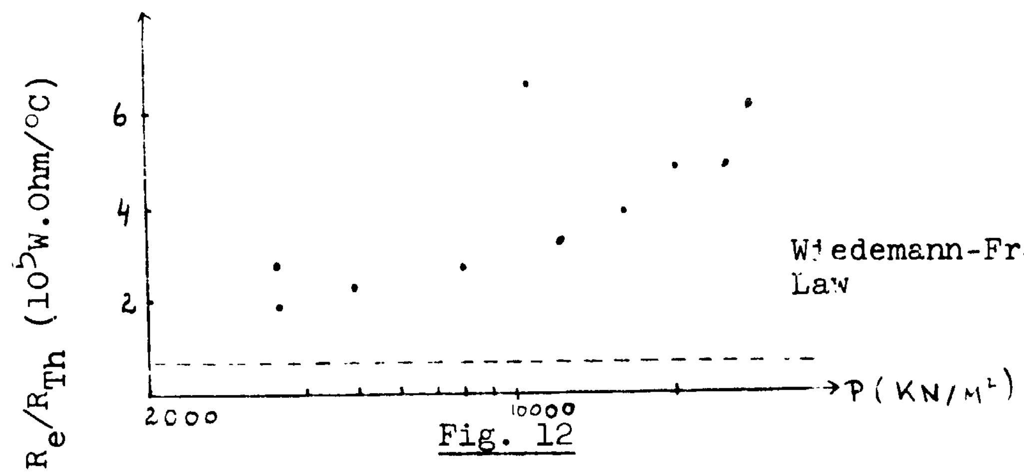

Fig. 12

If the mechanisms for both electrical and thermal conductivity were the same, one would expect the ratio R_E/R_{Th} to be equal to a constant

$$R_e/R_{Th} = 7.35 \; 10^{-6} \; \frac{\text{Ohm Watt}}{{}^{\circ}\text{C}} \tag{39}$$

(at a temperature $T = 27^{\circ}C$).

Actually the ratios are always higher than the one given by Wiedemann-Franz law (39). This indicates that the resistance stemming from the surface film is the dominant one as far as electrical conductivity is concerned, while it can probably be neglected for thermal conductivity. (There is, however, no clear proof of this).

If both constriction resistance and surface film resistance are important, the basic equation (6) for the heat flow has to be changed to

$$h_c = \frac{A^T}{\frac{1}{2ka} + \frac{b}{\pi a^2 k_s}} \tag{40}$$

since the constriction resistance $(\frac{1}{2ka})$ and the surface film resistance $(\frac{b}{k_s a^2})$ have to be added to one another. In (40) k_s is the thermal conductivity, b the thickness of the surface film.

Similarly the electrical current through one contact element becomes

$$J = \frac{V}{\frac{1}{2\sigma a} + \frac{b}{\pi a^2 \sigma_s}} \tag{41}$$

where V is the voltage across the interface; σ and σ_s are the electrical conductivities of metal and surface film, respectively. If the effects of the surface film were small the ratio of electrical/thermal resistance

$$R_e/R_{th} = \frac{\dfrac{1}{2\sigma a} + \dfrac{b}{\pi a^2 \sigma_s}}{\dfrac{1}{2ka} + \dfrac{b}{\pi a^2 k_s}} \approx k/\sigma \tag{42}$$

would be given by the Wiedemann-Franz law. The experimental evidence is, however, that $R_\epsilon/R_{th} >> {}^{k}/\sigma$ and a function of pressure. This indicates that surface films are dominant as far as the electrical conductivity is concerned, while the constriction resistance is the important one for heat transfer. Neglecting the appropriate terms in (40), (41), (42) we have

$$h_c = 2ka \ \Delta \ T \tag{43}$$

$$J = \frac{\pi a^2 \sigma_s V}{b} \tag{44}$$

$$R_e/R_{th} = \frac{2}{\pi} \left(\frac{b}{a}\right) \frac{k}{\sigma_s} \tag{45}$$

Since the force F and the electrical current J are both proportional to a^2 (for inelastic contacts), we obtain

$$J = \text{const. } F \tag{46}$$

which agrees well with experiment [8].

9. The High Pressure Region

In the high pressure region $(p \geq 10^{-3}\ p_c)$ the approximation (6) for the constriction resistance is not applicable anymore and a more accurate expression has to be used. A particularly simple relation is the one given by Cetinkale and Fishenden [9]

$$h_c = \frac{\pi k a \quad \Delta T}{\tan^{-1}(\frac{B}{a} - 1)} \tag{47}$$

For $B \gg a$ this agrees with the expression discussed before (6). If, however, $B \approx a$, i.e. if the contact points become large then

$$h_c = \frac{\pi k a^2 \cdot \Delta T}{B - a} \tag{48}$$

Equation (45) has to be changed accordingly. It becomes

$$R_e/R_{th} = \frac{k}{\sigma_s} \frac{1}{B - a} \tag{49}$$

The strong dependence of (48) on a is the probable reason for the striking increase of the heat flow at high pressures. However, more work has to be done before quantitative predictions about the behavior of the heat flow in the high pressure region can be made.

ACKNOWLEDGMENTS

We would like to thank Dr. Behram Kurşunoğlu and the University of Miami for the hospitality at the Center for Theoretical Studies. Furthermore, we thank Dr. Nejat Veziroğlu of the Dept. of Mechanical Engineering at the University of Miami for many valuable discussions.

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